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On the galactic rotation curves problem within an axisymmetric approach

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ABSTRACT

In Nucamendi et al. and Lake, it has been shown that galactic potentials can be kinematically linked to the observed red/blue shifts of the corresponding galactic rotation curves under a minimal set of assumptions: the emitted photons come from stable time-like circular geodesic orbits of stars in a static spherically symmetric gravitational field, and propagate to us along null geodesics. It is remarkable that this relation can be established without appealing at all to a concrete theory of gravitational interaction. Here, we generalize this kinematical spherically symmetric approach to the galactic rotation curves problem to the stationary axisymmetric realm since this is precisely the symmetry that spiral galaxies possess. Thus, by making use of the most general stationary axisymmetric metric, we also consider stable circular orbits of stars that emit signals which travel to a distant observer along null geodesics and express the galactic red/blue shifts in terms of three arbitrary metric functions, clarifying the contribution of the rotation as well as the dragging of the gravitational field. This stationary axisymmetric approach distinguishes between red and blue shifts emitted by circularly orbiting receding and approaching stars, respectively, even when they are considered with respect to the centre of a spiral galaxy, indicating the need of precise measurements in order to confront predictions with observations. We also point out the difficulties one encounters in the attempt of determining the metric functions from observations and list some potential strategies to overcome them.

Key words: Galaxy: disc – Galaxy: kinematics and dynamics – galaxies: distances and redshifts – galaxies: kinematics and dynamics – galaxies: spiral.

1 INTRODUCTION

The galactic rotation curves provide a direct method of determining the gravitational field inside a spiral galaxy since they have been measured for a great amount of galaxies (Rubin, Ford & Thonnard 1980; Rubin et al. 1982, 1985; Rubin 1983; Persic, Salucci & Stel 1996; Sofue & Rubin 2001). These curves are obtained by measuring the red/blue shifts of light emitted from stars and from the 21 cm radiation from the neutral gas clouds. The observations show evidence that the red/blue shifts z, or equivalently, the tangential velocities of rotation v, remain constant or decay more slowly than the Keplerian behaviour ($v^2 \sim 1/r$) up to distances far beyond the luminous radius of these galaxies. By performing a naive Newtonian analysis of this effect, one deduces that the energy density of the galaxies decreases approximately as r^{-2} and hence, the mass of these bodies should increase as $m(r) \approx r$. Since the observed luminous galactic components do not produce this growing behaviour, a question arises: What is the reason of such an effect? Nowadays there is a strong belief that dark matter is responsible for it, being the major bounded constituent of galaxies and galaxy clusters [$\sim$25 per cent of the total energy density of our Universe (Komatsu et al. 2011)]. However, one can naturally ask whether this large unseen mass does not produce a relevant gravitational redshift. On the other hand, in Salucci (2001) it was shown for a large and complete sample of spiral galaxies that their luminous regions consist of stellar discs embedded in universal dark haloes of constant density independent of the galaxy properties; moreover, it was shown that the dark matter haloes made by the most likely dark particle are inconsistent with actual observations. Finally, alternative approaches to this problem like modifications of Newtonian dynamics based on these observations have also been developed (Milgrom 1983; Boehmer, Harko & Lobo 2008; Milgrom & Sanders 2008; Mendoza et al. 2011; Hernandez, Jimenez & Allen 2012). Here, following Nucamendi, Salgado & Sudarsky (2001), we shall analyse the problem on the basis of what it is directly observed: the red/blue shifts. This approach enables us to keep track of the effect of the underlying made assumptions and to be aware of when they are no longer valid.

There have been previous approaches to the galactic rotation curves problem that make use of stationary axisymmetric metrics within the framework of General Relativity (see the comoving dust

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solution of Cooperstock & Tieu 2005a,b, 2006, 2007, 2008; Carrick & Cooperstock 2012). Within these models it is claimed that the known data from galactic rotation curve can be described with at most relatively little extra matter by non-linear general relativistic effects in galactic dynamics. An important point within their approach is that even though fields and velocities are small in a galaxy, it is not consistent to describe the latter through a Newtonian approximation since there are non-linear contributions of non-negligible size coming from the Einstein equations. However, these models have received criticisms in several directions that point out to unphysical features like the need of additional exotic matter source in the galactic disc (Korzynski 2005) or infinite mass at large distances (Menzies & Mathews 2006), the presence of singularities when continuing the interior solution into a consistent exterior configuration (Zingg, Aste & Trautmann 2007), among others, as well as the inconsistency of the use of comoving frames with the condition of differential rotation (Cross 2006). An important observation upon this model was pointed out in Rakic & Schwarz (2008), namely the stationary axisymmetric metric of the comoving dust solution of Cooperstock & Tieu (2005a) does not possess the most general form since it has only three arbitrary functions instead of four; moreover, their assumed Weyl gauge ($W = r$) is not consistent with the Einstein equations since this metric function is not harmonic and hence, it does not belong to the most general class of stationary axisymmetric solutions, the Lewis–Papapetrou class, a circumstance that might be connected with the problems of the model.

The aim of this paper is to provide a stationary axisymmetric kinematical description of the galactic rotation curves problem since this is a more realistic symmetry compared to the spherical one when studying spiral galaxies: the most accepted composition of spiral galaxies indicates that its main mass constituent is concentrated in a thin disc with a central bulge which are surrounded by a spherical halo (Binney & Tremaine 2008). Thus, it is commonly accepted that the main aspects of the galactic dynamics can be approximately described by (rotating) thin galactic disc models. We make use of the most general stationary axisymmetric metric and express the galactic red/blue shifts measured by a distant observer, that can in principle be compared to observations provided by astronomers, in terms of three arbitrary metric functions. We also clarify the contribution of the rotation and the dragging of inertial frames due to the gravitational field. We further point out the difficulties we have when determining the metric functions from observations without making reference to any theory of gravitational interactions, and comment on some possible ways to overcome them.

2 STATIONARY AXISYMMETRIC ROTATION CURVES

We shall start by assuming that stars behave like test particles which follow time-like geodesics of a rotating axially symmetric space–time associated with galactic discs which possess such a symmetry. The most general line element for a space–time of this kind has the following form:

$$\mathrm{d}s^2 = -\mathrm{e}^{2\Phi}\mathrm{d}t^2 + Q^2\mathrm{d}r^2 + R^2\left[\mathrm{d}\theta^2 + \sin^2\theta\,(\mathrm{d}\varphi - W\,\mathrm{d}t)^2\right], \quad (1)$$

where Φ, Q, R and W are all functions of r and θ. We shall also consider two observers $\mathcal{O}_e$ and $\mathcal{O}_d$ with 4-velocities u_e^μ, u_d^μ, respectively. Observer $\mathcal{O}_e$ corresponds to the light emitter (i.e. to the stars placed at a point P_e of space–time) and $\mathcal{O}_d$ represents the detector at point P_d, which is located far away from the light emitter and is ideally located at $r \longrightarrow \infty$.

We further assume that stars move on the galactic plane and, thus, the polar angle can be fixed $\theta = \pi/2$, so that $u_e^\mu = (U^t, U^r, 0, U^\varphi)_e$, where $U^\mu = \dot{x}^\mu$ and the dot stands for derivation with respect to the proper time of the particle (star).

On the other hand, the 4-velocity of the detector located 'far away' from the source, in a stationary axisymmetric background, is given by $u_d^\mu = (U^t, U^r, 0, U^\varphi)_d$ in the language of the above-mentioned coordinates and conventions. In this expression the U^φ component of the 4-velocity u_d^μ accounts for the dragging of the observer at P_d due to the rotation of the galaxy. This effect must be taken into account when considering the measurement of red/blue shifts of light signals emitted in our own Galaxy or in galaxies 'close' to ours, for instance. However, if the studied red/blue shifts correspond to galaxies located 'far away' from us, so that we can neglect the dragging effect, then we can consider that the detector is static, i.e. that the $\mathcal{O}_d$'s 4-velocity is tangent to the static Killing field $\frac{\partial}{\partial t}$, and hence its 4-velocity is given by $u_d^\mu = (U^t, U^r, 0, 0)_d$. Later on we shall give a qualitative estimation of whether we can consider this ideal limit (when the dragging effect can be neglected) in terms of the contribution of the angular velocity of a given galaxy to the measured red/blue shift.

We further normalize the 4-velocity as usual ($u^\mu u_\mu = -1$), a condition which renders the following relation:

$$-1 = g_{tt}\left(U^t\right)^2 + g_{rr}\left(U^r\right)^2 + g_{\varphi\varphi}\,(U^\varphi)^2 + 2g_{t\varphi}U^tU^\varphi. \quad (2)$$

The stationary axisymmetric metric (1) possesses two commuting Killing vectors: the time-like $\varepsilon^\mu = (1, 0, 0, 0)$ and the rotational one $\psi^\mu = (0, 0, 0, 1)$. The corresponding conserved quantities are the energy and the angular momentum per unit of mass at rest of the test particle and read

$$E = -g_{\mu\nu}\varepsilon^\mu u^\nu = -\left(g_{tt}U^t + g_{t\varphi}U^\varphi\right), \quad (3)$$

$$L = g_{\mu\nu}\psi^\mu u^\nu = g_{\varphi\varphi}U^\varphi + g_{t\varphi}U^t. \quad (4)$$

These relations are useful to obtain the expressions for U^t and U^φ in terms of the metric and these conserved quantities:

$$U^t = \frac{Eg_{\varphi\varphi} + Lg_{t\varphi}}{g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}} = \frac{E - LW}{\mathrm{e}^{2\Phi}}, \quad (5)$$

$$U^\varphi = -\frac{Eg_{t\varphi} + Lg_{tt}}{g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}} = \frac{L}{R^2} + \frac{(E - LW)\,W}{\mathrm{e}^{2\Phi}}. \quad (6)$$

By introducing these 4-velocity components (5) and (6) in the line element (2) we get

$$g_{rr}\left(U^r\right)^2 = EU^t - LU^\varphi - 1 \quad (7)$$

or, after multiplying by $\mathrm{e}^{2\Phi}$, we equivalently have

$$\mathrm{e}^{2\Phi}Q^2\left(U^r\right)^2 + \mathrm{e}^{2\Phi} + \frac{L^2\mathrm{e}^{2\Phi}}{R^2} - (E - LW)^2 = \mathrm{e}^{2\Phi}Q^2\left(U^r\right)^2 + V_{\mathrm{eff}}(E, L, g_{\mu\nu}) = 0. \quad (8)$$

This equation resembles an energy conservation law for a non-relativistic particle with position-dependent mass moving in an effective potential that depends on E and L. This equation can be further reduced by considering that the orbits of stars are circular $U^r = 0$:

$$V_{\mathrm{eff}} = \mathrm{e}^{2\Phi} + \frac{L^2\mathrm{e}^{2\Phi}}{R^2} - (E - LW)^2 = 0. \quad (9)$$

On the other hand, the time-like geodesics that follow the stars orbiting around the galaxies can be written as

$$\frac{\mathrm{d}U^\mu}{\mathrm{d}\tau} + \Gamma^\mu_{\rho\sigma}U^\rho U^\sigma = 0. \quad (10)$$

By computing the corresponding Christoffel symbols for the space–time metric (1) and substituting their expressions into (10) we obtain that geodesic equations for U^t and U^φ are satisfied identically, while for the radial geodesic equation we get

$$\frac{\mathrm{d}U^r}{\mathrm{d}\tau} + \frac{Q'}{Q}\left(U^r\right)^2 + \frac{1}{2Q^2}\left[\left(\frac{E-LW}{\mathrm{e}^{2\Phi}}\right)^2\left(\mathrm{e}^{2\Phi}\right)' - \frac{L^2}{R^4}\left(R^2\right)' + 2L\left(\frac{E-LW}{\mathrm{e}^{2\Phi}}\right)W'\right] = 0, \tag{11}$$

where τ is the proper time of the particle and $'$ denotes partial derivatives with respect to r. By considering circular orbits we set again $U^r = 0$ and obtain the following relation:

$$-\left(E - LW\right)^2\left(\mathrm{e}^{-2\Phi}\right)' + \left(\frac{L^2}{R^2}\right)' + 2L\left(\frac{E-LW}{\mathrm{e}^{2\Phi}}\right)W' = 0. \tag{12}$$

It turns out that this expression can be easily integrated yielding

$$\frac{L^2}{R^2} - \frac{(E-LW)^2}{\mathrm{e}^{2\Phi}} = k, \tag{13}$$

where k is an arbitrary constant. However, by comparing this result to the relation (9), we see that it is in fact a fixed constant: $k = -1$. This situation also takes place for the null geodesics of photons emitted by stars and detected by an observer located at $r \longrightarrow \infty$: their geodesic equation is equivalent to the normalization condition of their 4-momentum.

Once we have deleted the kinetic energy component in (8), one gets the expression for the effective potential on which the star moves (9). Thus, by deriving it with respect to the radial coordinate and setting the result to zero, we basically impose the minimum condition on the effective potential, a necessary condition to have circular orbits for the stars. This condition leads to the following expression for E:

$$E = LW - \frac{\left(\mathrm{e}^{2\Phi} + \frac{L^2\mathrm{e}^{2\Phi}}{R^2}\right)'}{2LW'}, \tag{14}$$

whereas from equation (9) we get the following relation:

$$E = LW + \mathrm{e}^{\Phi}\sqrt{1 + \frac{L^2}{R^2}}. \tag{15}$$

From both of these equations we see that the second term in the right-hand side of (14) must be positive definite, leading to the following restriction:

$$\left(\mathrm{e}^{2\Phi} + \frac{L^2\mathrm{e}^{2\Phi}}{R^2}\right)' \gtrless 0, \qquad W' \lessgtr 0, \tag{16}$$

where the last two inequalities must be satisfied simultaneously in the sense that the numerator and the denominator of that term must have opposite sign for positive L.

Moreover, from (14) and (15) we obtain the expression for the angular momentum L:

$$L_\pm = \sqrt{\frac{(W')^2 \pm |W'|\sqrt{(W')^2 + \frac{2R'\left(\mathrm{e}^{2\Phi}\right)'}{R^3} - \frac{\left(\mathrm{e}^{2\Phi}\right)'\left(\ln\frac{\mathrm{e}^{\Phi}}{R}\right)'}{R^2}}}{\frac{2}{R^2}\left\{\left[\left(\frac{\mathrm{e}^{\Phi}}{R}\right)'\right]^2 - (W')^2\right\}}}, \tag{17}$$

where the following restrictions must be fulfilled in order to have a real angular momentum:

$$(i) \quad \left(W'\right)^2 \geq -\frac{2R'\left(\mathrm{e}^{2\Phi}\right)'}{R^3}, \tag{18}$$

equivalently

$$\left(W'\right)^2 \geq \left(\mathrm{e}^{2\Phi}\right)'\left[(R)^{-2}\right]'; \tag{19}$$

this condition makes the square root of the second term in the radicand's numerator real;

$$(ii) \qquad \left(W'\right)^2 \pm \left|W'\right|\sqrt{(W')^2 + \frac{2R'\left(\mathrm{e}^{2\Phi}\right)'}{R^3}} - \frac{\left(\mathrm{e}^{2\Phi}\right)'\left(\ln\frac{\mathrm{e}^{\Phi}}{R}\right)'}{R^2} \gtrless 0, \tag{20}$$

$$\left|\left(\frac{\mathrm{e}^{\Phi}}{R}\right)'\right| \gtrless \left|W'\right|, \tag{21}$$

where, again, the last two inequalities must have the same sign simultaneously.

On the other hand, for the energy E we have from (15):

$$E_\pm = \mathrm{e}^{\Phi}\sqrt{1 + \frac{(W')^2 \pm |W'|\sqrt{(W')^2 + \frac{2R'\left(\mathrm{e}^{2\Phi}\right)'}{R^3}} - \frac{\left(\mathrm{e}^{2\Phi}\right)'\left(\ln\frac{\mathrm{e}^{\Phi}}{R}\right)'}{R^2}}{2\left\{\left[\left(\frac{\mathrm{e}^{\Phi}}{R}\right)'\right]^2 - (W')^2\right\}}} + W\sqrt{\frac{(W')^2 \pm |W'|\sqrt{(W')^2 + \frac{2R'\left(\mathrm{e}^{2\Phi}\right)'}{R^3}} - \frac{\left(\mathrm{e}^{2\Phi}\right)'\left(\ln\frac{\mathrm{e}^{\Phi}}{R}\right)'}{R^2}}{\frac{2}{R^2}\left\{\left[\left(\frac{\mathrm{e}^{\Phi}}{R}\right)'\right]^2 - (W')^2\right\}}}. \tag{22}$$

Moreover, in order for the circular orbits of stars to be stable, we also need the second derivative of the effective potential to be positive:

$$\left[\mathrm{e}^{2\Phi}\left(1 + \frac{L^2}{R^2}\right)\right]'' + 2L\left(E - LW\right)W'' - 2L^2\left(W'\right)^2 > 0, \tag{23}$$

where L and E must be replace by their expressions (17) and (22), respectively. It is worth noticing that relations (17) and (22), as well as the condition (23), can be reduced to the results obtained for the static spherically symmetric approach reported in Nucamendi et al. (2001) and Lake (2004) when $\theta = \pi/2$, $W = 0$ and $R = r$ under a simple coordinate transformation. For instance, by taking $N = \mathrm{e}^{\Phi}$ we have

$$E^2 = \frac{N^2}{1 - r\partial_r N/N}, \quad L^2 = \frac{r^3\partial_r N/N}{1 - r\partial_r N/N}, \tag{24}$$

and

$$E = \frac{\mathrm{e}^{\Phi}}{\sqrt{1 - r\Phi'}}, \quad L = \frac{r\sqrt{r\Phi'}}{\sqrt{1 - r\Phi'}}, \tag{25}$$

which are precisely the expressions obtained by Nucamendi et al. (2001) and Lake (2004), respectively, for the energy and angular momentum.

It is well known that rotation curves of spiral galaxies are inferred from the red and blue shifts of the radiation emitted by stars that move in (nearly) circular orbits around the central region of the galaxy (Rubin et al. 1980, 1982, 1985) and that light signals travel on null geodesics with tangent 4-momentum k^μ. Here, we shall make the assumption that k^μ are restricted to lie in the galactic plane $\theta = \pi/2$, and we shall evaluate the frequency shift of a light signal emitted from a star in a circular orbit represented by $\mathcal{O}_e$ and detected by an observer represented by $\mathcal{O}_d$. Moreover, we shall suppose that the galactic disc is edge-on directed towards the observer, a fact which implies that $k^\theta = 0$.[1]

The frequency shift associated with the emission and detection of light signals is given by

$$1 + z = \frac{\omega_e}{\omega_d} \quad , \tag{26}$$

where the frequency of a photon measured by an observer with proper velocity $u^\mu_C|_{P_C}$ reads

$$\omega_C = -k_\mu u^\mu_C|_{P_C} \quad , \tag{27}$$

and the index C refers to the emission (e) or detection (d) at the corresponding space–time point P_C. When the observer is comoving with the particle we have

$$\omega_e = -k_\mu u^\mu_e \quad . \tag{28}$$

Thus, a photon emitted at point P_C in the galactic plane possesses a 4-momentum $k^\mu_C = (k^t, k^r, 0, k^\varphi)_C$. The corresponding conserved quantities along the null geodesics of light signals read

$$E_\gamma = -g_{\mu\nu}\varepsilon^\mu k^\nu = -\left(g_{tt}k^t + g_{t\varphi}k^\varphi\right) \quad , \tag{29}$$

$$L_\gamma = g_{\mu\nu}\psi^\mu k^\nu = g_{\varphi\varphi}k^\varphi + g_{t\varphi}k^t \quad , \tag{30}$$

whereas the normalization condition for the 4-momentum $k^\mu k_\mu = 0$ is

$$0 = g_{tt}\left(k^t\right)^2 + g_{rr}\left(k^r\right)^2 + g_{\varphi\varphi}\left(k^\varphi\right)^2 + 2g_{t\varphi}k^t k^\varphi, \tag{31}$$

and leads to the following expression for the k^r in terms of the metric, the conserved quantities L_γ, E_γ, and k^t and k^φ:

$$g_{rr}\left(k^r\right)^2 = E_\gamma k^t - L_\gamma k^\varphi = \frac{g_{\varphi\varphi}E_\gamma^2 + 2g_{t\varphi}E_\gamma L_\gamma + g_{tt}L_\gamma^2}{g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}}. \tag{32}$$

There are two frequency shifts which correspond to the maximum and minimum values of ω_e associated with light propagation in the same and the opposite direction of the motion of the signal emitter, in other words, the frequency shifts of a receding and an approaching star, respectively. As we shall see later, these maximum/minimum values of the frequency shifts are reached for stars whose position vector $\boldsymbol{r}$, with respect to the galactic centre, is perpendicular to the detector's line of sight, i.e. along the plane where $k^r = 0$ for the observer (Lake 2004).

From the constant character along the geodesics of the product of the Killing vector field ε^μ with a geodesic tangent (29), together with the frequency definition (27) and taking into account that at ($r \longrightarrow \infty$), the 4-velocity of the detector is given by $u^\mu_d = (U^t, 0, 0, U^\varphi)_d$, we get the following expressions for the detector's frequency

$$\begin{aligned}\omega_d &= -k_\mu u^\mu_d \\ &= -\left(g_{tt}U^t k^t + g_{t\varphi}U^t k^\varphi + g_{\varphi t}U^\varphi k^t + g_{\varphi\varphi}U^\varphi k^\varphi\right)\big|_d \\ &= \left(E_\gamma U^t - L_\gamma U^\varphi\right)\big|_d \quad ,\end{aligned} \tag{33}$$

[1] This fact is also taken into account by astronomers when reporting the corresponding total blue or red shifts since they subtract the contribution coming from the inclination of the galactic disc from the plane $\theta = \pi/2$.

and the emitter's frequency

$$\begin{aligned}\omega_e &= -k_\mu u^\mu\big|_e \\ &= -\left(g_{tt}U^t k^t + g_{t\varphi}U^t k^\varphi + g_{\varphi t}U^\varphi k^t + g_{\varphi\varphi}U^\varphi k^\varphi\right. \\ &\quad \left.+ g_{rr}U^r k^r\right)\big|_e = \left(E_\gamma U^t - L_\gamma U^\varphi - g_{rr}U^r k^r\right)\big|_e .\end{aligned} \tag{34}$$

Therefore, with the aid of the frequency shift (26) we have for arbitrary star orbits

$$1 + z = \frac{\omega_e}{\omega_d} = \frac{\left(E_\gamma U^t - L_\gamma U^\varphi - g_{rr}U^r k^r\right)\big|_e}{\left(E_\gamma U^t - L_\gamma U^\varphi\right)\big|_d} \quad . \tag{35}$$

Since the star orbits we are considering are circular, then

$$1 + z = \frac{\omega_e}{\omega_d} = \frac{\left(E_\gamma U^t - L_\gamma U^\varphi\right)\big|_e}{\left(E_\gamma U^t - L_\gamma U^\varphi\right)\big|_d} = \frac{U^t_e - bU^\varphi_e}{U^t_d - bU^\varphi_d} \quad , \tag{36}$$

where we have introduced $b \equiv \frac{L_\gamma}{E_\gamma}$ – the impact parameter at infinity, and, hence, $|b|$ represents the radial distance at any side of the observed centre of the galaxy, where $b = 0$. Since we shall consider red/blue shifts either side of the central value $b = 0$, it is convenient to express (36) as

$$1 + z_\epsilon = \frac{\left(U^t - \epsilon|b|U^\varphi\right)\big|_e}{\left(U^t - \epsilon|b|U^\varphi\right)|_d} \quad , \tag{37}$$

where $\epsilon = \pm 1$, and to compute the following quantity:

$$1 + z_c = \frac{U^t_e}{U^t_d} = \frac{\left(\frac{E - LW}{e^{2\Phi}}\right)\big|_e}{\left(\frac{E - LW}{e^{2\Phi}}\right)\big|_d} \, , \tag{38}$$

where L and E are given by (17) and (22), respectively. This is the gravitational redshift of the centre of the galaxy measured by an observer located at $r \longrightarrow \infty$. Since in order to consider red/blue shifts either side of the central value $b = 0$, we need to subtract this quantity from (37), we define red and blue shifts as follows

$$Z_{\rm red} \equiv z_+ - z_c = \frac{\left(U^t_e U^\varphi_d - U^t_d U^\varphi_e\right)}{U^t_d\left(U^t_d - |b|U^\varphi_d\right)}|b| \quad , \tag{39}$$

$$Z_{\rm blue} \equiv z_c - z_- = \frac{\left(U^t_e U^\varphi_d - U^t_d U^\varphi_e\right)}{U^t_d\left(U^t_d + |b|U^\varphi_d\right)}|b| \quad , \tag{40}$$

corresponding to receding and approaching stars, respectively. It is obvious that now $Z_{\rm red} \neq Z_{\rm blue}$ (since $z_+ - z_c \neq z_c - z_-$). These quantities are the general red and blue shifts, respectively, experimented by light signals travelling along null geodesics and emitted by circularly orbiting stars around the centre of a galaxy to a distant observer. It is worth mentioning that in this formula we have dropped the gravitational redshift of the centre of the galaxy, a fact that it is indeed taken into account by astronomers when reporting their observed data.

In the special case in which the detector is located 'far away' enough from the source of information, i.e. if the contribution of the dragging of the detector's inertial frame due to the rotation of the system, encoded in U^φ_d, is negligible in comparison to the contribution coming from the U^t_d component, in other words, if $U^\varphi_d \ll U^t_d$, then the detector can be considered static at $r \longrightarrow \infty$ and its 4-velocity will be given by $u^\mu_d = e^{2\Phi(\infty)}\delta^\mu_t$. This fact can be understood as well from the following analysis: since we are considering the limit in which $U^\varphi_d \ll U^t_d$ and taking into account that $u^\mu_d = \frac{{\rm d}x^\mu}{{\rm d}s}|_d$, then

$$\frac{U^\varphi_d}{U^t_d} = \frac{{\rm d}\varphi}{{\rm d}t} \equiv \Omega_{\rm d} \ll 1, \tag{41}$$

where s is the proper time of the orbiting particle (star) and $\Omega_{\rm d}$ is the angular velocity measured by the detector at $r \longrightarrow \infty$.

Thus, in the special case in which $U_d^\varphi \ll U_d^t$ (or $\Omega_{\rm d} \ll 1$,) the general red/blue shifts (39) and (40) reduce to the effective (quasi-static) red/blue shift defined by the following expression:

$$Z = -\frac{U_e^\varphi}{U_d^t}|b| = -\frac{\left[\frac{L}{R^2} + \frac{(E-LW)W}{e^{2\Phi}}\right]\Big|_e}{\left(\frac{E-LW}{e^{2\Phi}}\right)\Big|_d}|b|\,, \tag{42}$$

which is symmetric with respect to the centre of the galaxy as in Nucamendi et al. (2001) and Lake (2004). It is worth noticing that even when the dragging of the detector's inertial frame is neglected, the differential rotation encoded in W, still plays a non-trivial role in the relation (42). Hence, we have

$$Z^2 = \frac{\left[\frac{L}{R^2} + \frac{(E-LW)W}{e^{2\Phi}}\right]^2\Big|_e}{\left(\frac{E-LW}{e^{2\Phi}}\right)^2\Big|_d} b^2\,. \tag{43}$$

We further need to take into account the light bending due to the gravitational field generated by the rotating galaxy, in other words, we need to construct a mapping between the impact parameter b and the location of the star r given by its vector position $\boldsymbol{r}$ with respect to the centre of the galaxy, i.e. the mapping $b(r)$. Following Nucamendi et al. (2001) and Lake (2004), we shall choose the maximum value of Z^2 at a fixed distance from the observed centre of the galaxy (at a fixed b). From (43) it follows that if the function factor that multiplies b^2 is monotone decreasing with increasing r, then the maximum observed value of Z^2 corresponds to the minimum value of r along the null geodesic of the photons. This minimum value of r corresponds to the position of the star on either side of the centre of the galaxy, lying on the plane perpendicular to the detector's line of sight, i.e. on the plane where $k^r = 0$ for the observer located at $r \longrightarrow \infty$. Thus, from (43) it follows that the squared impact parameter b^2 must also be maximized; this quantity can be calculated from the geodesic equation of the photons (or, equivalently, from the $k^\mu k_\mu = 0$ relation taking into account that $k^r = 0$) and is given by

$$b_\pm = \frac{-g_{t\varphi} \pm \sqrt{g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}}}{(g_{tt})} = -\frac{R^2W \pm Re^\Phi}{e^{2\Phi}},$$
$$b_\pm^2 = \frac{\left(R^2W \pm Re^\Phi\right)^2}{e^{4\Phi}}. \tag{44}$$

Since we look for the maximum value of b, then we should choose either b_+ or b_-, depending on the sign and magnitude of the product RW with respect to e^Φ, in such a way that b is maximum. Finally, the monotone decreasing condition of the factor that multiplies b^2 in (43) imposes the following restriction:

$$\left[\ln\left(\frac{L}{R^2} + \frac{(E-LW)W}{e^{2\Phi}}\right)_e\right]' < \left[\ln\left(\frac{E-LW}{e^{2\Phi}}\right)_d\right]'. \tag{45}$$

Thus, the mapping $b(r)$, responsible for the gravitational light bending, is given by (44) under the condition (45).

We should also mention here that the relations (44) and (45) reduce to the results obtained by Nucamendi et al. (2001) and Lake (2004) when $W = 0$ and $R = r$:

$$b^2 = \frac{r^2}{e^{2\Phi}} \qquad \text{and} \qquad \left(\ln\frac{L}{r^2}\right)' < \left(\ln\frac{E}{e^{2\Phi}}\right)', \tag{46}$$

respectively.

As mentioned above, observations are reported as

$$Z = v(b) - v(b=0), \tag{47}$$

where Z is given by (43), supplemented by the light bending mapping (44).

If one succeeds in determining the three unknown metric functions Φ, R and W from observational data, this would imply that the dynamics of light signals is determined by the geodesics of a stationary axisymmetric metric, independently of the assumption of the dynamics of the geometry (of a theory of gravitational interactions) or of the nature of dark matter, in the case that the latter is needed. However, the task of solving for Φ, R and W is not trivial at all compared to the spherically symmetric case considered in Nucamendi et al. (2001) and Lake (2004) where there was just one unknown metric function to be determined.

A way of determining the above-mentioned three arbitrary metric functions from observations consists in making use of the empiric Persic and Salucci's universal formula for rotation curves in the halo region of a galaxy (Persic et al. 1996), see also Salucci et al. (2007):

$$Z_i = \frac{\alpha_i b^2}{b^2 + \beta_i}, \tag{48}$$

where the constants α_i and β_i correspond to the description of the central, red and blue shifts defined in (38)–(40) and are determined from observations of a given galaxy. An alternative way to proceed is to consider just the red and blue shifts, (39) and (40), related to this formula and make use of the relation (14) [or (15) if possible] in order to complete the system of equations for Φ, R and W. Thus, in principle, we can conform a system of three non-linear differential equations of first order for three unknown metric functions. From the expressions found for the red/blue shifts, it is obvious that the difficulties that arise in solving such a system are not trivial at all.

There is another kind of universal rotation curve formula for flat spiral galaxies that can be adjusted to observed data (González, Plata-Plata & Ramos-Caro 2010) and can be used to determine Φ, R and W upon setting it to the red/blue shift language. Moreover, both of these universal rotation curve formulas could be used in a combined way in order to determine the metric functions Φ, R and W from observations for one and the same sample of galaxies.

One more way to find these functions consists of approaching the galactic rotation curves problem in the post-Newtonian approximation in the spirit of Ramos-Caro, Agón & Pedraza (2012), where the authors constructed analytical models that allow one to compute the first general relativistic corrections to the matter distributions and gravitational potentials for stationary systems with axial symmetry. It is worth mentioning that the main modifications in their approach appear far from the galaxy cores, a result that seems to be consistent with the predictions of Cooperstock & Tieu (2005a,b, 2006, 2007, 2008).

We would like to point out that one can make a combined use of the red/blue shifts of galactic rotation curves and gravitational lensing as proposed in Bharadwaj & Kar (2003) and (Faber & Visser 2006). However, as stated in Faber & Visser (2006), till now the combined measured data come from different distance scales (redshifts): most high-quality rotation curves are available for galaxies with a low to intermediate redshift of upto $z \sim 0.4$, while gravitational lenses are easier to detect at intermediate to high redshifts $z \gtrsim 0.4$. Therefore, both kinds of data are available for the same galaxy, but at different radii and, hence, they are not comparable. Thus, it is still difficult to take advantage from both sets of observations simultaneously. This situation will improve in the future when observations with a higher resolution will be carried out. In summary, more precise observational data are needed in order to benefit from the stationary axisymmetric approach to the galactic rotation curves problem within this context.

Finally, the stationary axisymmetric formalism presented here could be applied to a wider range of phenomena like binary systems, accretion discs of rotating black holes and active galactic nuclei where the size of the effects would be less restrictive.

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